Moments of Integrated Work in the M/G/1

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This note gives a straightforward way to compute the moments of the distribution of the integrated amount of work over the course of a busy period in an M/G/1. This quantity is the area under the work function, over a busy period. This problem was posed by Iglehart [3], with further work by Cohen [1] and Glynn et al. [2].

Consider an M/G/1 with arrival rate λ and job size distribution with random variable S. Let the state of the M/G/1 be (a, w), where a is the integrated work in the busy period so far, and w is the work in the system. During an idle period, a will remain at its value at the end of the previous busy period, resetting to 0 at the beginning of the next busy period.

Let A and W be the time-average values of a and w, and let A_r be the value of A at reset points.

The goal of this note is to determine the distribution of A_r. Specifically, we will calculate all of the moments of A_r.

1 Drift Method

There are four events in the system that can affect the drift: The deterministic increase in a, the deterministic decrease in w, arrivals, and resets at the beginning of busy periods.

w decreases at rate 1, with stochastic jumps of size S at rate λ. a increases at rate w, resetting to 0 at the beginning of each busy period.

We can thus write down the drift for an arbitrary test function $f(a, w)$. Let G denote the instantaneous generator of the M/G/1 system:

$$\begin{aligned} G \circ f(a, w) = w\frac{\partial f(a, w)}{\partial a} - \frac{\partial f(a, w)}{\partial w} + \lambda(E[f(a, w + S)] - f(a, w))\mathbb{1}\{w > 0\} \\ - \lambda(f(a, 0) - E[f(0, S)])\mathbb{1}\{w = 0\} \end{aligned}$$

Then, we can take expectation and apply the fact that $E[G \circ f(A, W)] = 0$.

To determine $E[A_r^k]$ for all k, we will use the drift method with test functions $a^k w^\ell$, $k, \ell \geq 0$. There are three cases to split things into: $k = 0$, $\ell = 0$, $k, \ell > 0$.

1.1 Area only ($\ell = 0$)

First, we compute the drift:

$$G \circ a^k = ka^{k-1}w - \lambda a^k \mathbb{1}\{w = 0\}$$

Next, we take expectations:

$$\begin{aligned} 0 &= kE[A^{k-1}W] - \lambda(1-\rho)E[A_r^k] \\ E[A_r^k] &= \frac{kE[A^{k-1}W]}{\lambda(1-\rho)} \end{aligned}$$

Note that we make use of the fact that arrivals are Poisson, and the PASTA principle, in this step. This ensures that $E[A \mid W = 0]$ and $E[A_r]$ are equal.

Thus, to compute moments of A_r, it suffices to compute expectations of multivariate polynomials of A and W.

1.2 Area and Work ($k, \ell > 0$)

First, we compute the drift:

$$\begin{aligned} G \circ a^k w^\ell &= ka^{k-1}w^{\ell+1} - \ell a^k w^{\ell-1} + \lambda a^k (E[(w+S)^\ell] - w^\ell) \\ &= ka^{k-1}w^{\ell+1} - \ell a^k w^{\ell-1} + \lambda a^k \sum_{i=1}^{\ell} \binom{\ell}{i} E[S^i] w^{\ell-i} \\ &= -(1-\rho)\ell(a^k w^{\ell-1}) + ka^{k-1}w^{\ell+1} + \lambda a^k \sum_{i=2}^{\ell} \binom{\ell}{i} E[S^i] w^{\ell-i} \end{aligned}$$

Next, we take expectations:

$$\begin{aligned} 0 &= -(1-\rho)\ell E[A^k W^{\ell-1}] + kE[A^{k-1}W^{\ell+1}] + \lambda \sum_{i=2}^{\ell} \binom{\ell}{i} E[S^i] E[A^k W^{\ell-i}] \\ E[A^k W^{\ell-1}] &= \frac{kE[A^{k-1}W^{\ell+1}] + \lambda \sum_{i=2}^{\ell} \binom{\ell}{i} E[S^i] E[A^k W^{\ell-i}]}{\ell(1-\rho)} \end{aligned}$$

Note that the expectation of a multivariate polynomial of A and W can be expressed in terms of expectations of multivariate polynomials with either the same total degree and lower degree of A, or lower total degree and the same degree of A.

Thus, to compute these expectations, it suffices to compute moments of W.

1.3 Work only ($k = 0$)

First, we compute the drift:

$$\begin{aligned} G \circ w^\ell &= -\ell w^{\ell-1} + \lambda \sum_{i=1}^{\ell} \binom{\ell}{i} E[S^i] w^{\ell-i} \\ &= -\ell(1-\rho) w^{\ell-1} + \lambda \sum_{i=2}^{\ell} \binom{\ell}{i} E[S^i] w^{\ell-i} \end{aligned}$$

Next, we take expectations:

$$\begin{aligned} 0 &= -\ell(1-\rho) E[W^{\ell-1}] + \lambda \sum_{i=2}^{\ell} \binom{\ell}{i} E[S^i] E[W^{\ell-i}] \\ E[W^{\ell-1}] &= \frac{\lambda \sum_{i=2}^{\ell} \binom{\ell}{i} E[S^i] E[W^{\ell-i}]}{\ell(1-\rho)} \end{aligned}$$

Note that the moments of W are expressed in terms of lower-degree moments of W and the moments of S. Thus, we can express the moments of A_r in terms of the moments of the size S.

2 Examples

To demonstrate the method, I will compute the first, second, and third moments of A_r, given in (1), (2), (3).

2.1 First moment

$$\begin{aligned} E[A_r] &= \frac{E[W]}{\lambda(1-\rho)} \\ E[W] &= \frac{\lambda E[S^2]}{2(1-\rho)} \\ E[A_r] &= \frac{E[S^2]}{2(1-\rho)^2} \qquad (1) \end{aligned}$$

Note that (1) matches Glynn et al. [2]'s equation (20), for which they cite Iglehart [3]'s original paper, for which I believe the relevant result is Lemma 2.4(c), though I'm not certain – the paper uses very different notation. This also matches Cohen [1]'s equation (3.4).

2.2 Second moment

$$
\begin{aligned}
E[A_r^2] &= \frac{2E[AW]}{\lambda(1-\rho)} \\
E[AW] &= \frac{E[W^3] + \lambda E[S^2]E[A]}{2(1-\rho)} \\
E[A] &= \frac{E[W^2]}{1-\rho} \\
E[A_r^2] &= \frac{E[W^3]}{\lambda(1-\rho)^2} + \frac{E[S^2]E[A]}{(1-\rho)^2} = \frac{E[W^3]}{\lambda(1-\rho)^2} + \frac{E[W^2]E[S^2]}{(1-\rho)^3}
\end{aligned}
$$

$$
\begin{aligned}
E[W^2] &= \frac{3\lambda E[S^2]E[W] + \lambda E[S^3]}{3(1-\rho)} = \frac{\lambda^2 E[S^2]^2}{2(1-\rho)^2} + \frac{\lambda E[S^3]}{3(1-\rho)} \\
E[W^3] &= \frac{6\lambda E[S^2]E[W^2] + 4\lambda E[S^3]E[W] + \lambda E[S^4]}{4(1-\rho)} \\
&= \frac{3\lambda^3 E[S^2]^3}{4(1-\rho)^3} + \frac{\lambda^2 E[S^2]E[S^3]}{(1-\rho)^2} + \frac{\lambda E[S^4]}{4(1-\rho)} \\
E[A_r^2] &= \frac{5\lambda^2 E[S^2]^3}{4(1-\rho)^5} + \frac{4\lambda E[S^2]E[S^3]}{3(1-\rho)^4} + \frac{E[S^4]}{4(1-\rho)^3} \qquad (2)
\end{aligned}
$$

Note that (2) matches Cohen [1]'s equation (3.9). Note that it does not match Glynn et al. [2]'s equation (21), despite that paper citing Cohen [1] as its source for the equation. I believe this is a transcription error – the 4 in the denominator of the $E[S^2]^3$ term and the 3 in the denominator of the $E[S^2]E[S^3]$ term are missing in Glynn et al. [2].

2.3 Third moment

$$
\begin{aligned}
E[A_r^3] &= \frac{3E[A^2W]}{\lambda(1-\rho)} \\
E[A^2W] &= \frac{2E[AW^3] + \lambda E[S^2]E[A^2]}{2(1-\rho)} \\
E[A^2] &= \frac{2E[AW^2]}{1-\rho} \\
E[AW^3] &= \frac{E[W^5] + 6\lambda E[S^2]E[AW^2] + 4\lambda E[S^3]E[AW] + \lambda E[S^4]E[A]}{4(1-\rho)} \\
E[AW^2] &= \frac{E[W^4] + 3\lambda E[S^2]E[AW] + \lambda E[S^3]}{3(1-\rho)}
\end{aligned}
$$

$$
\begin{aligned}
E[A_r^3] &= \frac{3E[AW^3]}{\lambda(1-\rho)^2} + \frac{3E[S^2]E[A^2]}{2(1-\rho)^2} \\
&= \frac{3E[W^5]}{4\lambda(1-\rho)^3} + \frac{15E[S^2]E[AW^2]}{2(1-\rho)^3} + \frac{3E[S^3]E[AW]}{(1-\rho)^3} + \frac{3E[S^4]E[A]}{4(1-\rho)^3} \\
&= \frac{3E[W^5]}{4\lambda(1-\rho)^3} + \frac{5E[W^4]E[S^2]}{2(1-\rho)^4} + E[AW]\left(\frac{15\lambda E[S^2]^2}{2(1-\rho)^4} + \frac{3E[S^3]}{(1-\rho)^3}\right) \\
&+ \frac{3E[S^4]E[A]}{4(1-\rho)^3} \\
&= \frac{3E[W^5]}{4\lambda(1-\rho)^3} + \frac{5E[W^4]E[S^2]}{2(1-\rho)^4} + E[W^3]\left(\frac{15\lambda E[S^2]^2}{4(1-\rho)^5} + \frac{3E[S^3]}{2(1-\rho)^4}\right) \\
&+ E[A]\left(\frac{15\lambda^2 E[S^2]^3}{4(1-\rho)^5} + \frac{3\lambda E[S^3]E[S^2]}{2(1-\rho)^4} + \frac{3E[S^4]}{4(1-\rho)^3}\right) \\
E[A_r^3] &= \frac{3E[W^5]}{4\lambda(1-\rho)^3} + \frac{5E[W^4]E[S^2]}{2(1-\rho)^4} + E[W^3]\left(\frac{15\lambda E[S^2]^2}{4(1-\rho)^5} + \frac{3E[S^3]}{2(1-\rho)^4}\right) \\
&+ E[W^2]\left(\frac{15\lambda^2 E[S^2]^3}{4(1-\rho)^6} + \frac{3\lambda E[S^3]E[S^2]}{2(1-\rho)^5} + \frac{3E[S^4]}{4(1-\rho)^4}\right) \\
E[W^5] &= \frac{15\lambda E[S^2]E[W^4] + 20\lambda E[S^3]E[W^3] + 15\lambda E[S^4]E[W^2] + 6\lambda E[S^5]E[W] + \lambda E[S^6]}{6(1-\rho)} \\
E[W^4] &= \frac{10\lambda E[S^2]E[W^3] + 10\lambda E[S^3]E[W^2] + 5\lambda E[S^4]E[W] + \lambda E[S^5]}{5(1-\rho)}
\end{aligned}
$$

$$
\begin{aligned}
E[A_r^3] &= \frac{35E[W^4]E[S^2]}{8(1-\rho)^4} + E[W^3]\left(\frac{15\lambda E[S^2]^2}{4(1-\rho)^5} + \frac{4E[S^3]}{(1-\rho)^4}\right) \\
&+ E[W^2]\left(\frac{15\lambda^2 E[S^2]^3}{4(1-\rho)^6} + \frac{3\lambda E[S^3]E[S^2]}{2(1-\rho)^5} + \frac{21E[S^4]}{8(1-\rho)^4}\right) + \frac{3E[W]E[S^5]}{4(1-\rho)^4} + \frac{E[S^6]}{8(1-\rho)^4} \\
&= E[W^3]\left(\frac{25\lambda E[S^2]^2}{2(1-\rho)^5} + \frac{4E[S^3]}{(1-\rho)^4}\right) \\
&+ E[W^2]\left(\frac{15\lambda^2 E[S^2]^3}{4(1-\rho)^6} + \frac{41\lambda E[S^3]E[S^2]}{4(1-\rho)^5} + \frac{21E[S^4]}{8(1-\rho)^4}\right) \\
&+ E[W]\left(\frac{35\lambda E[S^4]E[S^2]}{8(1-\rho)^5} + \frac{3E[S^5]}{4(1-\rho)^4}\right) + \frac{35\lambda E[S^5]E[S^2]}{8(1-\rho)^5} + \frac{E[S^6]}{8(1-\rho)^4} \\
&= E[W^2]\left(\frac{45\lambda^2 E[S^2]^3}{2(1-\rho)^6} + \frac{65\lambda E[S^3]E[S^2]}{4(1-\rho)^5} + \frac{21E[S^4]}{8(1-\rho)^4}\right) \\
&+ E[W]\left(\frac{25\lambda^2 E[S^3]E[S^2]^2}{2(1-\rho)^6} + \frac{4\lambda E[S^3]^2}{(1-\rho)^5} + \frac{35\lambda E[S^4]E[S^2]}{8(1-\rho)^5} + \frac{3E[S^5]}{4(1-\rho)^4}\right) \\
&+ \frac{25\lambda^2 E[S^4]E[S^2]^2}{8(1-\rho)^6} + \frac{\lambda E[S^4]E[S^3]}{(1-\rho)^5} + \frac{35\lambda E[S^5]E[S^2]}{8(1-\rho)^5} + \frac{E[S^6]}{8(1-\rho)^4}
\end{aligned}
$$

$$E[A_r^3] = E[W]\left(\frac{45\lambda^3 E[S^2]^4}{2(1-\rho)^7} + \frac{115\lambda^2 E[S^3]E[S^2]^2}{4(1-\rho)^6} + \frac{4\lambda E[S^3]^2}{(1-\rho)^5} + \frac{7\lambda E[S^4]E[S^2]}{(1-\rho)^5} + \frac{3E[S^5]}{4(1-\rho)^4}\right)$$
$$+ \frac{15\lambda^3 E[S^3]E[S^2]^3}{2(1-\rho)^7} + \frac{65\lambda^2 E[S^3]^2 E[S^2]}{12(1-\rho)^6} + \frac{25\lambda^2 E[S^4]E[S^2]^2}{8(1-\rho)^6}$$
$$+ \frac{15\lambda E[S^4]E[S^3]}{8(1-\rho)^5} + \frac{35\lambda E[S^5]E[S^2]}{8(1-\rho)^5} + \frac{E[S^6]}{8(1-\rho)^4}$$
$$E[A_r^3] = \frac{45\lambda^4 E[S^2]^5}{4(1-\rho)^8} + \frac{175\lambda^3 E[S^3]E[S^2]^3}{8(1-\rho)^7} + \frac{89\lambda^2 E[S^3]^2 E[S^2]}{12(1-\rho)^6} + \frac{53\lambda^2 E[S^4]E[S^2]^2}{8(1-\rho)^6}$$
$$+ \frac{15\lambda E[S^4]E[S^3]}{8(1-\rho)^5} + \frac{19\lambda E[S^5]E[S^2]}{4(1-\rho)^5} + \frac{E[S^6]}{8(1-\rho)^4} \tag{3}$$

To the best of my knowledge (and that of Glynn et al. [2]), (3) has not previously appeared in the literature.

I will stop here, but such formulas for arbitrary moments can be computed by this method with the aid of a computer algebra system.

References

[1] JW Cohen. Properties of the process of level crossings during a busy cycle of the M/G/1 queueing system. *Mathematics of Operations Research*, 3(2): 133–144, 1978.

[2] Peter W Glynn, Royi Jacobovic, and Michel Mandjes. Moments of polynomial functionals in Lévy-driven queues with secondary jumps. *arXiv preprint arXiv:2310.11137*, 2023.

[3] Donald L Iglehart. Functional limit theorems for the queue GI/G/1 in light traffic. *Advances in Applied Probability*, 3(2):269–281, 1971.