Moments of Integrated Work in the M/G/1

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This note gives a straightforward way to compute the moments of the distribution of the integrated amount of work over the course of a busy period in an M/G/1. This quantity is the area under the work function, over a busy period. This problem was posed by Iglehart [3], with further work by Cohen [1] and Glynn et al. [2].

Consider an M/G/1 with arrival rate $\lambda$ and job size distribution with random variable $S$. Let the state of the M/G/1 be $(a, w)$, where $a$ is the integrated work in the busy period so far, and $w$ is the work in the system. During an idle period, $a$ will remain at its value at the end of the previous busy period, resetting to 0 at the beginning of the next busy period.

Let $A$ and $W$ be the time-average values of $a$ and $w$, and let $A_r$ be the value of $A$ at reset points.

The goal of this note is to determine the distribution of $A_r$. Specifically, we will calculate all of the moments of $A_r$.

1 Drift Method

There are four events in the system that can affect the drift: The deterministic increase in $a$, the deterministic decrease in $w$, arrivals, and resets at the beginning of busy periods.

$w$ decreases at rate 1, with stochastic jumps of size $S$ at rate $\lambda$. $a$ increases at rate $w$, resetting to 0 at the beginning of each busy period.

We can thus write down the drift for an arbitrary test function $f(a, w)$. Let $G$ denote the instantaneous generator of the M/G/1 system:

\[
G \circ f(a, w) = w \frac{\partial f(a, w)}{\partial a} - \frac{\partial f(a, w)}{\partial w} + \lambda(E[f(a, w + S)] - f(a, w))\mathbb{1}\{w > 0\} - \lambda(f(a, 0) - E[f(0, S)])\mathbb{1}\{w = 0\}
\]

Then, we can take expectation and apply the fact that $E[G \circ f(A, W)] = 0$.

To determine $E[A_r^k]$ for all $k$, we will use the drift method with test functions $a^k w^\ell$, $k, \ell \geq 0$. There are three cases to split things into: $k = 0$, $\ell = 0$, $k, \ell > 0$. 

1.1 Area only ($\ell = 0$)

First, we compute the drift:

\[
G \circ a^k = ka^{k-1}w - \lambda a^k 1\{w = 0\}
\]

Next, we take expectations:

\[
0 = kE[A^{k-1}W] - \lambda (1 - \rho)E[A_r^k]
\]

\[
E[A^k] = \frac{kE[A^{k-1}W]}{\lambda(1 - \rho)}
\]

Note that we make use of the fact that arrivals are Poisson, and the PASTA principle, in this step. This ensures that $E[A | W = 0]$ and $E[A_r]$ are equal.

Thus, to compute moments of $A_r$, it suffices to compute expectations of multivariate polynomials of $A$ and $W$.

1.2 Area and Work ($k, \ell > 0$)

First, we compute the drift:

\[
G \circ a^k w^\ell = ka^{k-1}w^{\ell+1} - \ell a^k w^{\ell-1} + \lambda a^k (E[(w+S)\ell] - w^\ell)
\]

\[
= ka^{k-1}w^{\ell+1} - \ell a^k w^{\ell-1} + \lambda a^k \sum_{i=1}^{\ell} \binom{\ell}{i} E[S^i] w^{\ell-i}
\]

\[
= -(1 - \rho)\ell(a^k w^{\ell-1}) + ka^{k-1}w^{\ell+1} + \lambda a^k \sum_{i=2}^{\ell} \binom{\ell}{i} E[S^i] w^{\ell-i}
\]

Next, we take expectations:

\[
0 = -(1 - \rho)\ell E[A^k W^{\ell-1}] + kE[A^{k-1}W^{\ell+1}] + \lambda \sum_{i=2}^{\ell} \binom{\ell}{i} E[S^i] E[A^k W^{\ell-i}]
\]

\[
E[A^{k-1}W^{\ell+1}] = \frac{kE[A^{k-1}W^{\ell+1}] + \lambda \sum_{i=2}^{\ell} \binom{\ell}{i} E[S^i] E[A^k W^{\ell-i}]}{\ell(1 - \rho)}
\]

Note that the expectation of a multivariate polynomial of $A$ and $W$ can be expressed in terms of expectations of multivariate polynomials with either the same total degree and lower degree of $A$, or lower total degree and the same degree of $A$.

Thus, to compute these expectations, it suffices to compute moments of $W$.  

1.3 Work only \((k = 0)\)

First, we compute the drift:

\[
G \circ w^\ell = -\ell w^{\ell-1} + \lambda \sum_{i=1}^{\ell} \binom{\ell}{i} E[S^i] w^{\ell-i}
\]

\[
= -\ell (1 - \rho) w^{\ell-1} + \lambda \sum_{i=2}^{\ell} \binom{\ell}{i} E[S^i] w^{\ell-i}
\]

Next, we take expectations:

\[
0 = -\ell (1 - \rho) E[W^{\ell-1}] + \lambda \sum_{i=2}^{\ell} \binom{\ell}{i} E[S^i] E[W^{\ell-i}]
\]

\[
E[W^{\ell-1}] = \frac{\lambda \sum_{i=2}^{\ell} \binom{\ell}{i} E[S^i] E[W^{\ell-i}]}{\ell(1 - \rho)}
\]

Note that the moments of \(W\) are expressed in terms of lower-degree moments of \(W\) and the moments of \(S\). Thus, we can express the moments of \(A_r\) in terms of the moments of the size \(S\).

2 Examples

To demonstrate the method, I will compute the first, second, and third moments of \(A_r\), given in (1), (2), (3).

2.1 First moment

\[
E[A_r] = \frac{E[W]}{\lambda(1 - \rho)}
\]

\[
E[W] = \frac{\lambda E[S^2]}{2(1 - \rho)}
\]

\[
E[A_r] = \frac{E[S^2]}{2(1 - \rho)^2}
\]

Note that (1) matches Glynn et al. (2)'s equation (20), for which they cite Iglehart (3)'s original paper, for which I believe the relevant result is Lemma 2.4(c), though I'm not certain – the paper uses very different notation. This also matches Cohen (1)'s equation (3.4).
2.2 Second moment

\[
E[A_2^2] = \frac{2E[AW]}{\lambda(1-\rho)} \\
E[AW] = \frac{E[W^3] + \lambda E[S^2]E[A]}{2(1-\rho)} \\
E[A] = \frac{E[W^2]}{1-\rho} \\
E[A_2^2] = \frac{E[W^3]}{\lambda(1-\rho)^2} + \frac{E[S^2]E[A]}{(1-\rho)^2} = \frac{E[W^3]}{\lambda(1-\rho)^2} + \frac{E[W^2]E[S^2]}{(1-\rho)^3} \\
E[W^2] = \frac{3\lambda E[S^2]E[W] + \lambda E[S^3]}{3(1-\rho)} = \frac{\lambda^2 E[S^2]^2}{2(1-\rho)^2} + \frac{\lambda E[S^3]}{3(1-\rho)} \\
E[W^3] = \frac{6\lambda E[S^2]E[W^2] + 4\lambda E[S^3]E[W] + \lambda E[S^4]}{4(1-\rho)} \\
E[A_2^2] = \frac{5\lambda^2 E[S^2]^3}{4(1-\rho)^3} + \frac{4\lambda E[S^2]E[S^3]}{3(1-\rho)^4} + \frac{E[S^4]}{4(1-\rho)^5} \\
\tag{2}
\]

Note that (2) matches Cohen [1]'s equation (3.9). Note that it does not match Glynn et al. [2]'s equation (21), despite that paper citing Cohen [1] as its source for the equation. I believe this is a transcription error – the 4 in the denominator of the \(E[S^2]^3\) term and the 3 in the denominator of the \(E[S^2]E[S^3]\) term are missing in Glynn et al. [2].

2.3 Third moment

\[
E[A_3^3] = \frac{3E[A^2W]}{\lambda(1-\rho)} \\
E[A^2W] = \frac{2E[AW^3] + \lambda E[S^2]E[A^2]}{2(1-\rho)} \\
E[A^2] = \frac{2E[AW^2]}{1-\rho} \\
E[AW^2] = \frac{E[W^4] + 3\lambda E[S^2]E[AW] + \lambda E[S^3]}{3(1-\rho)}
\]
\[ E[\mathcal{A}]^2 = \frac{3E[AW^3]}{\lambda(1-\rho)^2} + \frac{3E[S^2]E[\mathcal{A}]^2}{2(1-\rho)^2} \]
\[ = \frac{3E[W^5]}{4\lambda(1-\rho)^3} + \frac{15E[S^2]E[AW^2]}{2(1-\rho)^3} + \frac{3E[S^3]E[AW]}{(1-\rho)^3} + \frac{3E[S^4]E[\mathcal{A}]}{4(1-\rho)^3} \]
\[ + \frac{5E[W^4]E[S^2]}{2(1-\rho)^3} + E[AW] \left( \frac{15\lambda E[S^2]^2}{2(1-\rho)^4} + \frac{3E[S^3]}{(1-\rho)^3} \right) \]
\[ + \frac{3E[S^4]E[\mathcal{A}]}{4(1-\rho)^3} \]
\[ = \frac{3E[W^5]}{4\lambda(1-\rho)^3} + \frac{5E[W^4]E[S^2]}{2(1-\rho)^4} + E[W^3] \left( \frac{15\lambda E[S^2]^2}{2(1-\rho)^5} + \frac{3E[S^3]}{(1-\rho)^4} \right) \]
\[ + E[\mathcal{A}] \left( \frac{15\lambda^2 E[S^2]^3}{4(1-\rho)^5} + \frac{3\lambda E[S^3]E[S^2]}{2(1-\rho)^4} + \frac{3E[S^4]}{4(1-\rho)^3} \right) \]
\[ E[\mathcal{A}]^3 = \frac{3E[W^5]}{4\lambda(1-\rho)^3} + \frac{5E[W^4]E[S^2]}{2(1-\rho)^4} + E[W^3] \left( \frac{15\lambda E[S^2]^2}{2(1-\rho)^5} + \frac{3E[S^3]}{(1-\rho)^4} \right) \]
\[ + E[W^2] \left( \frac{15\lambda^2 E[S^2]^3}{4(1-\rho)^6} + \frac{3\lambda E[S^3]E[S^2]}{2(1-\rho)^5} + \frac{3E[S^4]}{4(1-\rho)^4} \right) \]
\[ + \frac{3E[S^4]E[\mathcal{A}]}{4(1-\rho)^3} \]
\[ = \frac{35E[W^4]E[S^2]}{8(1-\rho)^4} + E[W^3] \left( \frac{15\lambda E[S^2]^2}{2(1-\rho)^5} + \frac{4E[S^3]}{8(1-\rho)^4} \right) \]
\[ + E[W^2] \left( \frac{25\lambda E[S^2]^3}{2(1-\rho)^5} + \frac{4E[S^3]}{8(1-\rho)^4} \right) \]
\[ + \frac{25\lambda^2 E[S^2]^4}{2(1-\rho)^5} + 4\lambda E[S^3]E[S^2] + 21E[S^4] \]
\[ + \frac{35\lambda E[S^4]E[S^2]}{8(1-\rho)^5} + \frac{3\lambda E[S^5]}{8(1-\rho)^4} + \frac{35\lambda E[S^5]E[S^2]}{8(1-\rho)^5} + \frac{E[S^6]}{8(1-\rho)^4} \]
\[ = E[W^3] \left( \frac{45\lambda E[S^2]^3}{2(1-\rho)^6} + \frac{65\lambda E[S^3]E[S^2]}{4(1-\rho)^5} + \frac{21E[S^4]}{8(1-\rho)^4} \right) \]
\[ + E[W] \left( \frac{25\lambda^2 E[S^2]^4}{2(1-\rho)^6} + \frac{6\lambda E[S^3]E[S^2]}{8(1-\rho)^5} + \frac{35\lambda E[S^4]E[S^2]}{8(1-\rho)^5} + \frac{3E[S^5]}{4(1-\rho)^4} \right) \]
\[ + \frac{25\lambda^2 E[S^2]^4}{8(1-\rho)^6} + \frac{\lambda E[S^4]E[S^2]}{8(1-\rho)^5} + \frac{35\lambda E[S^5]E[S^2]}{8(1-\rho)^5} + \frac{E[S^6]}{8(1-\rho)^4} \]
\[ E[A_3^2] = E[W] \left( \frac{45\lambda^3 E[S^2]^4}{2(1-\rho)^7} + \frac{115\lambda^2 E[S^3] E[S^2]^2}{4(1-\rho)^6} + \frac{4\lambda E[S^3]^2}{(1-\rho)^5} + \frac{7\lambda E[S^4] E[S^2]}{(1-\rho)^5} + \frac{3 E[S^5]}{4(1-\rho)^4} \right) \\
+ \frac{15\lambda^3 E[S^3] E[S^2]^3}{2(1-\rho)^7} + \frac{65\lambda^2 E[S^3]^2 E[S^2]}{12(1-\rho)^6} + \frac{25\lambda^2 E[S^4] E[S^2]^2}{8(1-\rho)^6} \\
+ \frac{15\lambda E[S^4] E[S^3]}{8(1-\rho)^5} + \frac{35\lambda E[S^5] E[S^2]}{8(1-\rho)^5} + \frac{E[S^6]}{8(1-\rho)^4} \\
E[A_3^5] = \frac{45\lambda^4 E[S^2]^5}{4(1-\rho)^8} + \frac{175\lambda^3 E[S^3] E[S^2]^3}{8(1-\rho)^7} + \frac{89\lambda^2 E[S^3]^2 E[S^2]}{12(1-\rho)^6} + \frac{53\lambda^2 E[S^4] E[S^2]^2}{8(1-\rho)^6} \\
+ \frac{15\lambda E[S^4] E[S^3]}{8(1-\rho)^5} + \frac{19\lambda E[S^5] E[S^2]}{4(1-\rho)^5} + \frac{E[S^6]}{8(1-\rho)^4} \tag{3} \]

To the best of my knowledge (and that of Glynn et al. [2]), (3) has not previously appeared in the literature.

I will stop here, but such formulas for arbitrary moments can be computed by this method with the aid of a computer algebra system.

References

