Perfect subsets of divisor multisets

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1 Result

We define a *divisor multiset* S of size n to be a multiset of positive integers such that $|S| = n$, and such that for all $s \in S$, s divides n.

For instance, if $n = 6$, then $S = [1, 1, 1, 2, 2, 3]$ is a divisor multiset.

Let's define a *perfect subset* S' of a divisor multiset S with $|S| = n$ to be a subset $S' \subset S$ such that the sum of the elements of S' is exactly n.

For instance, for the divisor multiset $S = [1, 1, 1, 2, 2, 3]$, the subset $S' = [1, 1, 1, 3]$ is a perfect subset. Two more perfect subsets of S are $[1, 1, 2, 2]$ and $[1, 2, 3]$.

Now, we are ready to state our main theorem:

Theorem 1. *Each divisor multiset S has a perfect subset S'.*

Proof. We will proceed via strong induction on n, the size of the divisor multiset.

As a base case, the only size-1 divisor multiset is [1]. It is a perfect subset of itself.

We must prove that if each divisor multiset of size $< n$ has a perfect subset, then each divisor multiset of size n has a perfect subset as well.

We will split into 3 cases:

- S contains at least $n/6$ 1s,
- n is of the form $2^i 3^j$ for integers i, j and S contains $< n/6$ 1s, or
- n has a prime factor $k \geq 5$ and S contains $< n/6$ 1s.

2 At least $n/6$ 1s

First, we will consider the case where S contains many copies of 1. Specifically, assume that the multiplicity of 1 in S, which we write $\#\{1 \in S\}$, is at least $n/6$.

In this case, we can directly construct the perfect subset S' via a simple algorithm.

Order the elements of S from largest to smallest, so that

$$S(1) \geq S(2) \geq \ldots \geq S(n).$$

Construct the sequence P_i of initial subsets of the ordering:

$$P_i := \{S(j) \mid j \leq i\}$$

Let i^* be the greatest index such that $\text{sum}(P_{i^*}) \leq n$. Note that $\text{sum}(S) \geq |S| = n$, so $i^* = n$ only if $\text{sum}(S) = n$. We will show that $\text{sum}(P_{i^*}) \geq 5n/6$.

Because $\#\{1 \in S\} \geq n/6$, we may construct a perfect subset by combining P_{i^*} with $n - \text{sum}(P_{i^*})$ copies of 1.

Note that if P_{i^*} contains a 1, then its last element is a 1, so $\text{sum}(P_{i^*}) = n$. Therefore, this construction does not double-count any 1s.

To prove that $\text{sum}(P_{i^*}) \geq 5n/6$, note that

$$n - \text{sum}(P_{i^*}) < S(i^* + 1) \leq S(i^*).$$

Therefore, to prove that $n - \text{sum}(P_{i^*}) \leq n/6$, we need only consider sequences of the i^* largest elements of S in which all elements are greater than $n/6$. We need only consider elements $n, n/2, n/3, n/4, n/5$.

We enumerate all such sequences here. We list i^* elements if $\text{sum}(P_{i^*}) = n$, and $i^* + 1$ elements otherwise. We write g_{i^*} as a shorthand for $n - \text{sum}(P_{i^*})$.

Sequence	g_{i^*}	Sequence	g_{i^*}
n	0	$n/2, n/2$	0
$n/2, n/3, n/3$	$n/6$	$n/2, n/4, n/4$	0
$n/2, n/4, n/5, n/5$	$n/20$	$n/2, n/5, n/5, n/5$	$n/10$
$n/3, n/3, n/3$	0	$n/3, n/3, n/4, n/4$	$n/12$
$n/3, n/3, n/5, n/5$	$2n/15$	$n/3, n/4, n/4, n/4$	$n/6$
$n/3, n/4, n/5, n/5, n/5$	$n/60$	$n/3, n/5, n/5, n/5, n/5$	$n/15$
$n/4, n/4, n/4, n/4$	0	$n/4, n/4, n/4, n/5, n/5$	$n/20$
$n/4, n/4, n/5, n/5, n/5$	$n/10$	$n/4, n/5, n/5, n/5, n/5$	3/20
$n/5, n/5, n/5, n/5, n/5$	0		

In all cases, $n - \text{sum}(P_{i^*}) \geq n/6$. As a result, if $\#\{1 \in S\} \geq n/6$, a perfect subset of the form P_{i^*} plus 1s must exist.

3 n of the form $2^i 3^j$

Suppose that n is of the form $2^i 3^j$, for some integers i and j, and that $\#\{1 \in S\} < n/6$.

Let S_2 be the set of even integers in S:

$$S_2 := \{s \mid s \in S, s \text{ is even}\}$$

Let S_r be the remaining integers in S:

$$S_r := \{s \mid s \in S, s \text{ is odd}, s > 1\}$$

Note that because 2 and 3 are the only prime factors of n, all elements of S_r are divisible by 3.

Because S is disjointly partitioned into 1s, S_2 and S_r, and $\#\{1 \in S\} < n/6$, $|S_2| + |S_r| > 5n/6$. As a result, it must be the case that either $|S_2| \geq n/2$, or $|S_r| \geq n/3$.

In the case that $|S_2| \geq n/2$, consider the set $S_2/2$:

$$S_2/2 := \{s/2 \mid s \in S, s \text{ is even}\}$$

Every element of $S_2/2$ is a factor of $n/2$, and $|S_2/2| \geq n/2$. Thus, within $S_2/2$ is a divisor multiset of size $n/2$. By the inductive hypothesis, this divisor multiset has a perfect subset summing to $n/2$. Multiplying each element by 2, we get a perfect subset of S itself.

If $|S_r| \geq n/3$, we can similarly construct $S_r/3$ and apply the inductive hypothesis to get a perfect subset of S.

4 n has a prime factor $k \geq 5$

Finally, suppose that n has a prime factor $k \geq 5$, and S contains $< n/6$ 1s.

Let us form the set S_k consisting of the elements of S which are multiples of k, and S_r consisting of the elements of S which are neither 1 nor multiples of k.

As in Section 3, if $|S_k| \geq n/k$, we can apply the inductive hypothesis to S_k/k to find a perfect subset of S.

If $|S_k| < n/k \leq n/5$, then

$$|S_r| = n - \#\{1 \in S\} - |S_k| \geq n - n/6 - n/5 = 19n/30.$$

Note that all elements of S_r are divisors of n/k, because they are divisors of n which are not multiples of k.

Because $19n/30 > n/5 \geq n/k$, we can apply the inductive hypothesis to a size-n/k subset of S_r, finding a perfect subset R_1 with sum n/k. Note that all elements of S_r have value at least 2, so $|R_1| \leq n/2k$.

Let us create the set $S_r^1 := S_r \setminus R_1$, where the superset indicates how many subsets have been removed.

We can apply this construction again to create $R_2, R_3, \ldots R_k$, all with sum n/k, as long as $S_r^1, S_r^2, \ldots, S_r^{k-1}$ have at least n/k elements. Because we remove at most $n/2k$ elements per iteration, we can lower bound the sizes of these sets

$$|S_r^i| \geq \frac{19n}{30} - i\frac{n}{2k}$$

In particular, we can lower bound the size of the final set:

$$|S_r^{k-1}| \geq \frac{19n}{30} - \frac{(k-1)n}{2k} = \frac{19n}{30} - \frac{n}{2} + \frac{n}{2k} = \frac{2n}{15} + \frac{n}{2k}$$

To prove that $|S_r^{k-1}| \geq n/k$, we just need to show that $2n/15 \geq n/2k$. But $k \geq 5$, so $2n/15 > n/10 \geq n/2k$.

Thus, we can always extract k disjoint perfect subsets of sum n/k from S_r using the inductive hypothesis. Combining these subsets, we form a perfect subset of S. □